CHAPTER 5

The analysis of the star catalogues' systematic errors

0. BASIC CONCEPTION

0.1. A demonstrative analogy

The necessity of analyzing the errors contained in star catalogues was already explained above. First and foremost, we are referring to the Almagest; however, the method in question shall also be applied to other catalogues – real ones as well as artificially generated ones. In the present chapter we shall demonstrate how to discover and compensate the systematic error. The idea behind the method is simple and quite natural. Moreover, it has been used in mathematical statistics for quite a while now. In order to explain the basic concept, let us consider the following example. Let us assume that we are regarding the results of a shooting competition as shown on the picture.

The dots represent bullet holes. How great is the hit accuracy? The answer is obvious – not that great at all. However, we can see that the actual grouping of shots is good enough. This leads us to the assumption that the rifleman is in fact a good one; as for the fact, that the bullets hit a spot which lies sideways from the bull's eye, it can be explained by a defect in his rifle-sight. Obviously, we can say nothing about the nature of said defect without seeing the rifle – however, we can estimate the displacement value. A sensible way of doing this would require us to determine the geometrical centre of all the results and draw a vector from the bull's eye to the calculated centre (vector S on the scheme). How do we formally calculate vector S? The procedure is a simple one. We have to take vectors x_i which correspond to the i^{th} result of the shooting and to average them by the total amount of shots N:

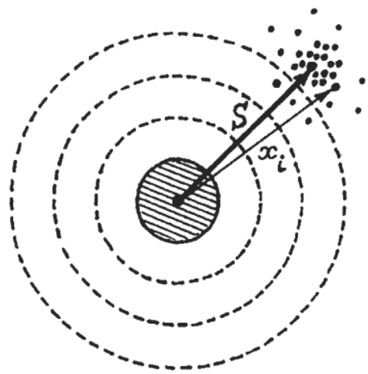


Fig. 5.0. A target with traces of bullet shots.

$$S = \frac{1}{N}\sum_{i=1}^{N} x_i \ .$$

We must also point out that vector S can be calculated alternatively from the problem of square average discrepancy minimization – we have to find vector S which provides for the minimum of the function

$$\sum_{i=1}^{N}(x_i - S)^2.$$

Here we estimate that $(x_i - S)^2 = (x_{i1} - S_1)^2 + (x_{i2} - S_2)^2$, where x_{i1}, x_{i2} and S_1, S_2 are the respective coordinates of vectors x_i and S.

The accuracy of the actual rifleman can then be characterized by the result scatter range around the discovered centre; this accuracy is thus a lot greater than the accuracy of hitting the bull's eye. The calculation of vector S represents the actual systematic error compensation procedure for this example (whose value equals S, respectively).

Formally, if we are to use a different coordinate system moving its initial point sideways from the bull's eye by vector S, the shooting results as given in the new coordinate system shall only contain random compounds (resulting from shaking hands etc), with no regular compound.

Let us now return to the star catalogue and assume that we need to check whether there may be a systematic error in some part of the catalogue and to determine its value should such an error indeed exist. Let us assume that we aren't confronted by the problem of dating so far – that is to say, we know the date when catalogue t_A was compiled for certain (A is for Almagest, of course – still, all the above considerations are valid for other catalogues as well). We would then have to compare the real coordinates of the stars for the moment t_A (known from precise modern catalogues) to the coordinate values taken from the catalogue under study which pertain to the part thereof that is used in our research. This comparison requires the calculation of the average discrepancy rate for the coordinates under comparison, just like we did in the example with rifle shot accuracy.

Let the total of stars from the chosen area equal N. We shall use the indications l_i and L_i for the actual ecliptic longitude of star i in the catalogue under study and its exact longitudinal value, respectively. In this case, the average (systematic) longitude error shall equal

$$\Delta\overline{L} = \frac{1}{N}\sum_{i=1}^{N}(l_i - L_i),$$

with the systematic latitudinal error equalling

$$\Delta\overline{B} = \frac{1}{N}\sum_{i=1}^{N}(b_i - B_i).$$

These errors, as we already mentioned, may result from the incorrect estimation of the ecliptic plane as well as a number of other reasons which remain unknown to us. We shall not be able to say anything in re the exact nature of these circumstances – however, we shall put forth a number of hypotheses in this respect. All of this notwithstanding, we can, and will, compensate the error that they caused. It requires nothing but the alteration of the catalogue coordinate system similarly to how it was done in the rifle example – one that would make the resultant average longitudinal and latitudinal errors equal zero.

0.2. The implementation of the method

In this section we shall demonstrate the practical application of the general concept related above.

First of all, let us emphasize that we shall only compensate the latitudinal error. The reasons were all named above – basically, it allows to minimize the error in calculations, which is vital, considering the low precision of the old catalogues.

Thus, what we have at our disposal is the catalogue from which we have selected a large group of stars whose total number equals N, with the coordinates $(l_i, b_i)_{i=1}^{N}$. Their doubles from the modern catalogue are already known to us from the previously conducted identification procedure. Let us use the indications $(L_i(t), B_i(t))_{i=1}^{N}$ for referring to the coordinates of said doubles calculated for moment t. Let us now assume that we want to examine the possible systematic error value under the assumption that the catalogue compilation date is t_A.

Let us define

$$L_i^A = L_i(t_A),\ B_i^A = B_i(t_A)$$

and introduce the latitudinal discrepancy

$$\Delta B_i^A = B_i^A - b_i.$$

Our goal is to minimize the value of

$$\sigma^2 = \sum_{i=1}^{N} (\Delta B_i^A)^2 \longrightarrow \sigma^2_{\min},$$

by changing the coordinate system, or simply drawing a new coordinate grid that differs from the one used in the catalogue.

The change of the coordinate grid can be parameterised by two values if we are to consider the problem of minimizing the expression mentioned above: γ and φ. They can be seen in fig. 5.1 below. Let us explain what they stand for. Here γ is the angle between the real ecliptic and the ecliptic of the catalogue, whereas φ represents the angle between the equinox line and the line of intersection between the real ecliptic and the catalogue ecliptic.

Thus, having solved the problem of minimizing the abovementioned expression, we can calculate the values of γ_{stat} and φ_{stat} which can parameterise the coordinate system alteration and give us the initial minimum. Their explicit form can be seen below, in formulae 5.5.2 and 5.5.3.

The value of σ_{min} is a residual square average latitudinal error that we end up after the compensation of the systematic error. The explicit form of the residual dispersion formula σ_{min} can be seen below, after formula 5.5.10. It results from using γ_{stat} and φ_{stat} as the parameters for the square average aberration expression. The derivation of these formulae can be seen below.

However, we cannot presume to have found the systematic error (or, rather, the parameters γ_{stat} and φ_{stat} that characterize it) with absolute precision. The matter is that individual measurement errors (which are of a random nature) also affect the values of γ_{stat} and φ_{stat}. Therefore, we can only claim that the real values of the systematic error are close to γ_{stat} and φ_{stat}.

In order to make our statement more precise, let us introduce the concept of a "trusted interval". Let 1-ε stand for a certain level of trust. If $\varepsilon = 0.1$, for instance, the level of trust shall equal 0.9. The level of trust represents the probability that guarantees the precision of our results; the trusted interval is the interval that includes the unknown real value of the parameter with a minimal probability of 1-ε. Let us define

$$I_\gamma(\varepsilon) = [\gamma_{stat} - x_\varepsilon, \gamma_{stat} + x_\varepsilon]$$

(or the trusted interval for the real value of parameter γ), and

$$I_\varphi(\varepsilon) = [\varphi_{stat} - y_\varepsilon, \varphi_{stat} + y_\varepsilon]$$

which is the trusted interval for the real value of the parameter φ. It can be demonstrated (qv below) that the values of x_ε and y_ε can be calculated by the formulae

$$x_\varepsilon = q_\varepsilon, y_\varepsilon = q_\varepsilon,$$

where q_ε represents $(1-\frac{\varepsilon}{2})$ – the fractile of the standard normal distribution as calculated from the tables.

Thus, if we are to define a certain confidence level 1-ε, we can guarantee that the real value of γ falls into the interval $I_\gamma(\varepsilon)$, and the value of φ falls into the interval $I_\varphi(\varepsilon)$ with a probability of no less than 1-ε.

0.3. The value of the systematic error cannot be used for the dating of the catalogue

Let us now provide a somewhat different interpretation of the calculated values of γ_{stat} and φ_{stat}. The use of stellar coordinates (it suffices to consider nothing but the latitudes, as a matter of fact) permits an easy calculation of the ecliptic poles P_A (for the catalogue under study) and $P(t)$ for the calculation catalogue of the moment t, qv in the diagram.

It is obvious that the arc distance between P_A and $P(t)$ equals γ_{stat} precisely, and that the compensation

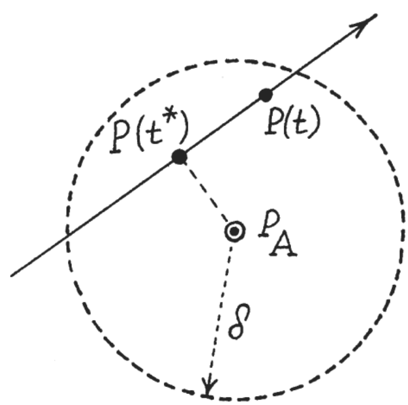


Fig. 5.0a. The two poles – on the ecliptic and in the catalogue.

of the systematic error requires nothing but the superposition of these two poles. Let us now consider the changes in the general picture that take place over the course of time. Since $P(t)$ shifts within the limits of one degree, we can use a flat diagram and assume that the motion of $P(t)$ is uniform, qv in the diagram.

Velocity v of this uniform motion is easy enough to calculate if we know the values of γ_{stat} for two different points. We can then calculate the moment t^* when the position of the real pole is the closest to that of the catalogue pole. Prima facie we might assume that this moment can be declared as the dating moment deduced from processing the coordinates of a great many stars. However, we have already demonstrated the fallacy of such logic; therefore, it has to be said that one cannot date the catalogue to the moment of t^*. Indeed, if the possible systematic error in Ptolemy's estimation of the ecliptic can equal the value of δ, all the moments in time that correspond to the passage of the pole $P(t)$ through a circle with the radius equalling δ whose centre lays in the point P_A should be regarded as possible candidates for the moment of dating. However, we do not know the value of δ. We can naturally estimate it, but only given that we know the dating of the catalogue. A different presumed dating shall yield a different estimation value. Therefore, this value already contains the presumed dating.

Thus, depending on Ptolemy's systematic error, or the error in the determination of the ecliptic, the moment t^* can either precede the real date of the catalogue's compilation or postdate it. In the former case, the catalogue (or, rather, the part of it for which we are trying to estimate the value of γ_{stat}), gains "extra age", beginning to resemble a catalogue compiled in the year t^*. In the latter case (when t^* postdates the real compilation dating) the catalogue becomes more recent. Below we shall see that both these possibilities are implemented in the Almagest. However, the terms "extra age" and "more recent" refer to a catalogue where the systematic errors were not compensated. What we end up with after the compensation is a "refined catalogue" which only contains random errors whose square average value can be estimated to equal σ_{min}, although no individual value can be determined.

Let us now consider the practical use of the general idea as specified above in more detail.

1. MAIN DEFINITION

From this chapter and on we shall assume to be dealing with a catalogue whose every star has a single double among the stars of the modern catalogue. Accordingly, we shall be using index i in order to identify the stars, as well as l_i and b_i for the ecliptic longitude and latitude of star i in the Almagest, respectively. $L_i(t)$ and $B_i(t)$ shall be used for referring to the real longitude and latitude of star i in epoch t. Bear in mind that time t is calculated backwards from 1900 A.D. and measured in centuries – that is to say, $t = 3.15$ shall correspond to the year $1900 - 3.15 \times 100 = 1585$ A.D., for instance, and $t = 22.0$ shall correspond to the year $1900 - 22 \times 100 = 300$ B.C.

Let t_A equal the unknown time of the Almagest catalogue compilation. The real longitude and latitude of star i for the year when the catalogue was compiled shall be indicated as L_i^A and B_i^A – that is, $L_i^A = L_i(t_A)$, $B_i^A = B_i(t_A)$. Let $\Delta B_i(t) = B_i(t) - b_i$ stand for the difference between the real latitude of star i for moment t and its latitude as given in the Almagest. The value of $\Delta B_i(t)$ shall be referred to as the latitudinal discrepancy for moment t. This value shall stand for the error in the estimation of the latitude of the Almagest star i under the condition that it was compiled in epoch t. It is natural that the real error in the estimation of the latitude is represented by $\Delta B_i(t_A) = \Delta B_i^A$.

As we already pointed out in Chapter 3, we only have to analyze the latitudinal errors in the case of the Almagest. The reasons for this were explained in detail above.

2. THE PARAMETERISATION OF GROUP ERRORS AND SYSTEMATIC ERRORS

Let us consider a certain group of stars such as a constellation or several constellations. We shall define the group error in the latitudinal coordinates of these stars as the error in the estimation of stellar latitudes for the group in question resulting from the motion of the stellar configuration under study across the celestial sphere as a whole. Therefore (we shall put a special emphasis on this circumstance due to its extensive use below), any subset of this configuration also shifts across the celestial sphere as a whole with the same angle as the entire configuration. Such shifts have three degrees of freedom – that is, they can be described by the specification of three parameters which we shall shortly define.

In fig. 5.1 one sees a diagram of the above. The position of the real ecliptic for the time moment t_A is represented on the celestial sphere whose centre is in point O. The respective points of the vernal and autumnal equinoxes are marked Q and R on the ecliptic. Point P represents North Pole of ecliptics. Point E represents the position of a given star. As we have already mentioned, all the group errors for a fixed stellar group in the ecliptic latitude made by the compiler of the catalogue can be considered to stem from the miscalculation of the ecliptic pole without exception, or the result of the fact that the compiler used the wrong point for the pole – P_A instead of P.

This point corresponds to the perturbed ecliptic which is referred to as the catalogue ecliptic in fig. 5.1. Its position can be determined in a unique way after we determine the following two parameters – firstly, angle γ between the lines OP and OP_A, or the very same plane angle between the planes of the real ecliptic and the catalogue ecliptic. Secondly, we must calculate angle φ between the equinox line RQ and line CD that results from the intersection of the real ecliptic plane with that of the catalogue ecliptic. This parameterisation is convenient for analytic purposes. However, we shall also be using value β alongside φ, which can be interpreted as follows (see fig. 5.1). The shift of the ecliptic can be decomposed into the composition of two rotations – one around the equinox axis RQ equalling angle γ, and the other around the axis that also lies within the plane of the ecliptic and is perpendicular to axis RQ and equals angle β. Thus, β stands for the length of arc Q_AQ which pertains to the large circumference that goes through pole P_A and point Q. The astronomical meaning of the point Q_A is clear enough. It is the vernal equinox point on the ecliptic of the catalogue. It is obvious that angles γ and φ unambiguously define the angles γ and β; the reverse is also true. The desired relation can be determined from the consideration of a spherical right-angled triangle CQ_AQ. The angle at the vertex Q_A is a right one, the angle at the vertex C equals γ, and the length of arc CQ equals β. The result is as follows:

$$\sin\beta = \sin\gamma\sin\varphi \qquad (5.2.1)$$

The third degree of freedom is defined by the rotation of the sphere around the axis $P_AP'_A$, qv in fig. 5.1. However, this rotation only affects stellar longitudes, leaving their latitudes intact. Therefore, we shall not be considering this degree of freedom. Let us point out that instead of the parameters specified we could choose any other set of basis parameters that define the rotation of the sphere. This obviously cannot affect the further conceptual development of our method.

Let us now study the distortion of the real coordinates of star i as affected by the systematic error of this kind. The real latitude B_i^A and the latitude of this star L_i^A are equal to the lengths of arcs EE' and QE' counted clockwise as seen from pole P, respectively. The respective distorted latitude and longitude b_i and l_i equal the lengths of arc EE_A and Q_AE_A. Bear in mind that the latitudes of stars whose real longitudes are greater than the latitude of point D and smaller than that of point C are reduced, whereas other latitudes increase, qv in fig. 5.1. This corollary does not apply to all stars, strictly speaking. It is false for the stars located at the angle distance of γ or less from the poles P and P'. However, since the value of γ is anything but great, there are very few stars which can be found in

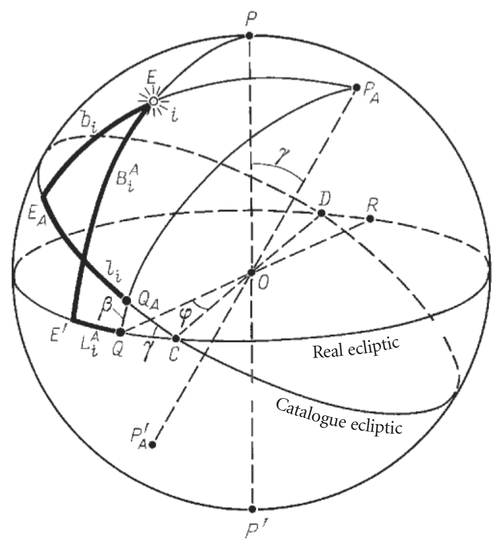


Fig. 5.1. Parameters defining the systematic error.

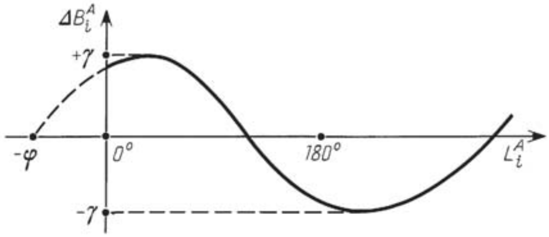


Fig. 5.2. The dependency between the systematic latitudinal discrepancy and the longitude.

such a small area. Virtually none of those are contained in the Almagest catalogue. As we shall see, the value of γ equals circa 20'.

Bearing in mind the value of γ being minute, one can suggest the following approximated formula for the latitudinal discrepancy:

$$\Delta B_i^A = \gamma \cdot \sin(L_i^A + \varphi). \qquad (5.2.2)$$

In other words, the systematic error in stellar latitude estimation can be represented with the sine curve we see in fig. 5.2. It is very much like the curve discovered earlier by Peters and Knobel ([1339]) when they were processing the data from the Almagest catalogue. The error rate of formula 5.5.2 does not exceed 1' for the stars whose $|b_A| \leq 80°$ and is therefore of no importance to us, so we shall consider the formula 5.2.1 absolutely precise. For the sake of propriety we shall exclude the stars whose absolute latitudinal values exceed 80 degrees from further consideration. We shall refer to the systematic error hereinafter, since the methods described are only valid under the assumption that we are considering a large group of stars. The verification of whether or not the discovered discrepancy coincides with group errors for individual constellations is a problem in itself. Its application to the Almagest is considered below, in Chapter 6.

Assuming that the time t_A of the catalogue's compilation is known, we can calculate the parameters γ and φ which define the systematic error as follows:

1) We shall calculate the real latitudes B_i^A and longitudes L_i^A for all the stars from the group under consideration (corresponding to the moment t_A).

2) Then we must find the values of parameters γ^* and φ^* which lead us to the solution of the problem in question.

$$\sigma^2(\gamma^*, \varphi^*) \to \min, \qquad (5.2.3)$$

where

$$\sigma^2(\gamma, \varphi) = \sum (B_i^A - b_i - \gamma \sin(L_i^A + \varphi))^2.$$

Had there been no other errors in the catalogue except for the systematic ones, the relation 5.2.3 would transform into the equation $\sigma^2(\gamma^*, \varphi^*) = 0$. However, the presence of random errors in stellar coordinates makes the minimum of 5.2.3 differ from zero.

In our situation, the catalogue compilation moment t_A remains unknown; therefore, we must calculate the systematic errors for all possible values of t from the interval $0 \leq t \leq 25$ under study, namely, the position of the real ecliptic and the equinox axis are calculated for every value of t. Then, just as we see it in fig. 5.1, the parameters $\gamma = \gamma(t)$, $\varphi = \varphi(t)$ and $\beta = \beta(t)$ are introduced; they define the relative positions of the catalogue ecliptic and the ecliptic for epoch t. The values of $\gamma(t)$ and $\varphi(t)$ are found as the solution of the problem

$$\sigma^2(\gamma(t), \varphi(t), t) \to \min, \qquad (5.2.4)$$

where

$$\sigma^2(\gamma, \varphi, t) = \sum (\Delta B_i(t) - \gamma \sin(L_i(t) + \varphi))^2. \quad (5.2.5)$$

Once again, had this case been ideal (with no other discrepancies but the systematic error inherent in the catalogue), the relation 5.2.4 could be transcribed as the following equation (disregarding the minute effects of proper star movement): $\sigma^2(\gamma(t), \varphi(t), t) = 0$.

As for the proper movement effects, let us remind the reader that the quantity of visibly mobile stars on the celestial sphere is very small as compared to the entire number of the Almagest stars. The solution of this last equation would exist for all the values of t; however, these equations would not enable us to calculate the date of t_A. It is all the more impossible to calculate it from the relation 5.2.4 which acts as a substitute for the equation in question when we consider a real catalogue containing random errors. We can merely calculate the systematic error as a function of the alleged dating t. This error is naturally dependent on the presumed dating due to the fluctuation of the ecliptic over the course of time. It is precisely why we aren't referring to the dating of the catalogue, but rather the deduction of its systematic error as a function of the alleged dating t.

The real catalogue contains random errors apart from the indicated systematic errors. Therefore, the discrepancies $B_i(t)$-B_i are random, and their values are scattered around the sine curve of their average value as seen in fig. 5.2. Assuming that other errors of the catalogue than the systematic ones are of a random nature, the problem of calculating $\gamma(t)$ and $\varphi(t)$ is one of regression parameter determination.

3. CALCULATING PARAMETERS $\gamma(t)$ AND $\varphi(t)$ WITH THE METHOD OF MINIMAL SQUARES

Let us find the solution for the minimization problem 5.2.4 and 5.2.5 expressed as $\gamma(t)$ and $\varphi(t)$. Below, in actual examples, this problem will be considered for groups containing different quantities of stars. We shall therefore be using the following standardized values for our calculations for which N will define the quantity of stars in the group under study.

$$\sigma_0^2(\gamma,\varphi,t)=\frac{1}{N}\sigma^2(\gamma,\varphi,t),$$

$$s_b(t)=\frac{1}{N}\sum_{i=1}^{N}\Delta B_i(t)\sin L_i(t),$$

$$c_b(t)=\frac{1}{N}\sum_{i=1}^{N}\Delta B_i(t)\cos L_i(t),$$

$$s_2(t)=\frac{1}{N}\sum_{i=1}^{N}\sin^2 L_i(t),$$

$$c_2(t)=\frac{1}{N}\sum_{i=1}^{N}\cos^2 L_i(t),$$

$$d(t)=\frac{1}{N}\sum_{i=1}^{N}\sin L_i(t)\cos L_i(t).$$

Let us point out that all such values can be calculated for any time moment t, depending on the values of the modern stellar coordinates as well as the star coordinates in the Almagest catalogue.

Obviously, the minimization problem 5.2.4 is equivalent to the minimization problem

$$\sigma_0^2(\gamma,\varphi,t)\to\min, \qquad (5.3.1)$$

in the sense that the parameters $\gamma(t)$ and $\varphi(t)$ defined by the relation 5.3.1 coincide with the parameters defined by the solution of the problem 5.2.4.

As we already pointed out, solving problem 5.3.1 only makes sense for large stellar groups, and since we shall study the statistical properties of such a solution below, we shall hereafter use $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ in order to refer to values which satisfy to relation 5.3.1.

The value of

$$\sigma_{min}(t)=\sigma_0(\gamma_{stat}(t),\varphi_{stat}(t),t) \qquad (5.3.2)$$

is rather transparent from the point of view of physics. It is the square average latitudinal discrepancy as applied to the group of stars under study for moment t resulting from the compensation of the discovered systematic error in $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$. As we shall see below, the value of $\sigma_{min}(t)$ is hardly dependent on time at all due to the extremely low proper movement velocity of most stars. Thus, we shall also use the indication σ_{min}. Bear in mind that the square average latitudinal discrepancy prior to the compensation of this error would equal the following value for moment t:

$$\sigma_{init}=\sigma_0(0,0,t)=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\Delta B_i(t))^2}, \qquad (5.3.3)$$

Thus, the difference $\Delta\sigma(t)=\sigma_{init}(t)-\sigma_{min}(t)$ estimates the effect of compensating the systematic error $\gamma_{stat}(t)$, $\varphi_{stat}(t)$.

Further on when we shall define the values of $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ from the relation 5.3.1, we shall presume the time moment t to be fixed. We shall therefore omit argument t from our calculations, that is, we shall use L_i instead of $L_i(t)$, s_b instead of $s_b(t)$ etc.

In order to find the minimum in the relation 5.3.1, we shall take the partial derivatives of functions $\sigma_0^2(\gamma,$

φ, t) by γ and φ and render them to zero. Bearing the formula $\sin(L_i + \varphi) = \sin L_i \cos\varphi + \cos L_i \sin\varphi$ in mind, we shall end up with the following equations:

$$s_b \cos\varphi + c_b \sin\varphi = \\ = \gamma \left[s_2 \cos^2\varphi + 2d \cos\varphi \sin\varphi + c_2 \sin^2\varphi\right], \qquad (5.3.4)$$

$$-c_b \cos\varphi + s_b \sin\varphi = \\ = \gamma \left[-d \cos^2\varphi + (s_2 - c_2) \cos\varphi \sin\varphi + d \sin^2\varphi\right]. \qquad (5.3.5)$$

If we divide the equation 5.3.4 by 5.3.5, we shall get

$$\frac{s_b + c_b \tan\varphi}{-c_b + s_b \tan\varphi} = \frac{s_2 + 2d\tan\varphi + c_2 \tan^2\varphi}{-d + (s_2 - c_2)\tan\varphi + d\tan^2\varphi}.$$

Once we render both parts of this equation to a common denominator, we shall come to the following equation concerning $\tan\varphi$:

$$(1+\tan^2\varphi)(c_b s_2 - s_b d) + (1+\tan^2\varphi)\tan\varphi(c_b d - s_b c_2) = 0.$$

This makes it easy to calculate the tangent of the optimal value of φ_{stat}:

$$\tan\varphi_{stat} = \frac{s_b d - c_b s_2}{c_b d - s_b c_2}. \qquad (5.3.6)$$

The equation 5.3.6 permits a unique determination of φ_{stat}; after that, the optimal value of γ_{stat} can be deducted from 5.3.4, for instance:

$$\gamma_{stat} = \\ = \frac{s_b \cos\varphi_{stat} + c_b \sin\varphi_{stat}}{s_2 \cos^2\varphi_{stat} + 2d\cos\varphi_{stat}\sin\varphi_{stat} + c_2 \sin^2\varphi_{stat}} = \\ = \frac{\sqrt{c_b d^2 - 2 s_b c_b d + s_b^2 c_2^2 + c_b^2 s_2^2}}{d^2 - s_2 c_2} \qquad (5.3.7)$$

Formulae 5.3.6 and 5.3.7 make it feasible to find the desired solution of the problem of calculating the estimations for φ_{stat} and γ_{stat} by the method of minimal squares.

It would be expedient to conduct a sensitivity analysis for this problem. Let us regard the second-order partial derivatives of the function $\sigma^2(\gamma, \varphi, t)$ with respect to γ and φ:

$$a_{11}(t) = \left.\frac{\partial^2\sigma^2(\gamma,\varphi,t)}{\partial\gamma^2}\right|_{\gamma=\gamma_{stat}(t),\varphi=\varphi_{stat}(t)}$$

$$a_{12}(t) = \left.\frac{\partial^2\sigma^2(\gamma,\varphi,t)}{\partial\gamma\partial\varphi}\right|_{\gamma=\gamma_{stat}(t),\varphi=\varphi_{stat}(t)}$$

$$a_{22}(t) = \left.\frac{\partial^2\sigma^2(\gamma,\varphi,t)}{\partial\varphi^2}\right|_{\gamma=\gamma_{stat}(t),\varphi=\varphi_{stat}(t)}$$

Keeping in mind the equations 5.3.4-5.3.7, we can easily determine the following expressions for these partial derivatives:

$$a_{11} = 2(s_2\cos^2\varphi_{stat} + 2d\cos\varphi_{stat}\sin\varphi_{stat} + c_2\sin^2\varphi_{stat}) = \\ = (2/\gamma_{stat})(s_b\cos\varphi_{stat} + c_b\sin\varphi_{stat}), \\ a_{12} = 2(c_b\cos\varphi_{stat} - s_b\sin\varphi_{stat}), \\ a_{22} = 2\gamma_{stat}^2(s_2\sin^2\varphi_{stat} - 2d\sin\varphi_{stat}\cos\varphi_{stat} + c_2\cos^2\varphi_{stat}). \qquad (5.3.8)$$

In order to estimate the errors in calculating the square average error rate $\sigma(\gamma, \varphi, t)$ considering the aberration of the values γ and φ from the calculated optimal values φ_{stat} and γ_{stat}, let us use the following decomposition of the function $\sigma^2(\gamma, \varphi, t)$ for the vicinity of point $(\gamma(t), \varphi(t))$:

$$\sigma^2(\gamma,\varphi,t) \approx \sigma^2_{\min} + a_{11}(t)(\gamma - \gamma_{stat}(t))^2 + \\ + 2a_{12}(t)(\gamma - \gamma_{stat}(t))(\varphi - \varphi_{stat}(t)) + a_{22}(t)(\varphi - \varphi_{stat}(t))^2. \qquad (5.3.9)$$

In the last formula we disregard the terms of magnitude order three and higher as related to the differences $\gamma - \gamma_{stat}(t)$ and $\varphi - \varphi_{stat}(t)$.

Formula (5.3.9) allows for an elementary estimation of the sensitivity of the square average error $\sigma(\gamma, \varphi, t)$ to the variation of parameters γ and φ. For this purpose it suffices to determine the values a_{11}, a_{12} and a_{22} pertinent to the right part of 5.3.9. After the estimation of $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$, they can be easily calculated by the formula 5.3.8.

Formula 5.3.9 demonstrates that the "level curves" of square average errors manifest as ellipses on the plane (γ, φ), qv in fig. 5.3. The centre of the ellipsis is in point $(\gamma_{stat}, \varphi_{stat})$ for which the value of the square average error equals σ_{min}. The direction of the ellip-

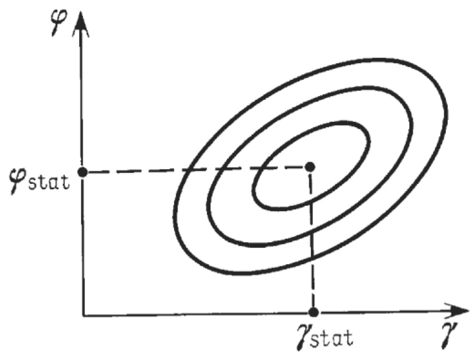


Fig. 5.3. Level curves of square average error σ (γ, φ, *t*) where *t* is a fixed value.

tic axes and the relation between them are determined by the standard analytical geometry formulae through the values a_{11}, a_{12} and a_{22}, namely, the tilt angle α of one of the ellipse axes is determined by the following relation:

$$\tan 2\alpha = \frac{2a_{12}}{a_{11} + a_{22}}.$$

The second axis is perpendicular to the first. The lengths of the axes relate to each other as λ_1/λ_2, where λ_1 and λ_2 are the roots of the quadratic equation

$$\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}\, a_{22} - a_{12}^2) = 0.$$

4. VARIATION OF THE PARAMETERS $\gamma_{stat}(t)$ AND $\varphi_{stat}(t)$ OVER THE COURSE OF TIME

Above we have made the assumption that the moment *t* is fixed. We shall now consider how the passage of time affects the behaviour of the calculated values γ_{stat} and φ_{stat}.

This behaviour can be determined from the formulae cited in the previous section. These formulae contain the values $L_i(t)$ and $B_i(t)$ which define the temporal dependency of γ_{stat} and φ_{stat}. The changes of the longitudes ($L_i(t)$) and the latitudes ($B_i(t)$) over the course of time have been studied well enough, qv in Chapter 1. The respective calculations were of a complex enough nature and required the use of a computer for a quantitative calculation of temporal dependency estimation for $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$, qv in Chapter 6. We shall merely analyse the qualitative behaviour of these functions herein.

Let us once again consider the celestial sphere, assuming all of the stars thereupon to be immobile for the sake of simplicity, thus returning to Ptolemy's conceptions, albeit merely for the sake of simplifying the argumentation and the calculations. We are well entitled to it since the percentage of the stars with noticeable proper movement velocity (ones that move by several arc minutes over the 2500-year time interval under study) is comparatively low. Such stars hardly affect the calculation of parameters $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ that we are concerned with presently.

In fig.5.4 one sees the celestial sphere as well as the real ecliptic for catalogue compilation epoch t_A. It would be expedient to compare figs. 5.1 and 5.4. In the epoch t_A that remains unknown to us the ecliptic pole $P(t_A)$ was occupying a certain position on the celestial sphere. The compiler of the catalogue was naturally not ideally precise in his indication of the ecliptic on the celestial sphere. Therefore the pole P_A of his "catalogue ecliptic" assumed a position differing from that of $P(t_A)$.

Let us draw the arc of a large circle that shall connect the pole $P(t_A)$ with the respective vernal and autumnal equinox points *Q* and *R*. In addition, we shall

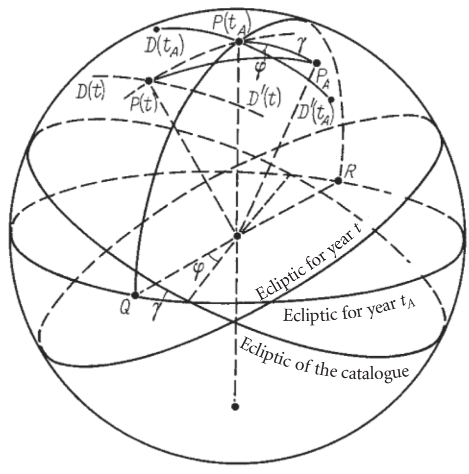


Fig. 5.4. Geometrical definition of the angles φ and γ on the celestial sphere.

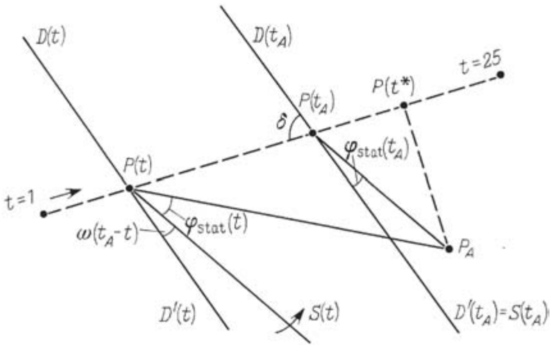


Fig. 5.5. Calculating the qualitative temporal dependency of $\gamma_{stat}(t)$ and $\varphi_{stat}(t_A)$.

draw the arc of the large circle $D(t_A)D'(t_A)$ that shall pass through $P(t_A)$ and cross the recently-built arc $Q\ P(t_A)R$ at a right angle in point $P(t_A)$. If we knew the date t_A, then the method of minimal squares as described in section 3, would give us the opportunity to find the parameters of γ and φ that define the mutual disposition of the ecliptic for epoch t_A and the catalogue ecliptic. Fig. 5.4 demonstrates that these very angles also define the mutual disposition of the poles $P(t_A)$ and P_A on the celestial sphere – namely, the value of γ equals the length of the arc $P(t_A)P_A$ in arc values, and angle φ equals the angle $P_AP(t_A)D'(t_A)$. As we point out in Chapter 1, the celestial position of the ecliptic alters in the course of time. This is the manifestation of the ecliptic fluctuation effect. Therefore, the ecliptic pole for moment t that differs from t_A shall be located in point $P(t)$ which will also differ from $P(t_A)$.

The ecliptic pole trajectory on the celestial sphere is indicated with a dotted line in fig. 5.4, one that crosses the points $P(t)$ and $P(t_A)$. Thus, in order to combine the ecliptic of epoch t with the catalogue ecliptic, one has to superpose poles P_A and $P(t)$ over one another. The length of arc $P(t)P_A$ equals the value of $\gamma_{stat}(t)$, and the location of the ecliptic rotation axis that provides for such a superposition can be parameterised by angle $P_AP(t)D'(t)$ where arc $D(t)D'(t)$ is "parallel" to arc $D(t_A)D'(t_A)$.

In order to understand the quantitative behaviour of the functions $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ better, let us use a two-dimensional drawing depicting just the ecliptic pole shifts. This is permissible, since the values of their shifts are a priori known to fall into the range of one degree. We shall thus make a two-dimensional copy of the polar part of fig. 5.4 – see fig. 5.5.

As it is obvious from fig. 5.5, the real ecliptic pole shifts over the course of time due to the fluctuation of the ecliptic. For the interval under study the value of this shift equals a mere 25' on the average, and so we can draw it as a straight line (see the dotted line in fig. 5.5). The motion of the ecliptic pole along this line can be considered uniform with a high enough degree of precision. And so, the distance between the poles $P(t)$ and $P(t_A)$, for example, shall equal $v(t_A - t)$ where v is the velocity of the ecliptic pole's motion. This velocity approximates 0.01' per year. As we have mentioned above, in the observation epoch t_A the catalogue compiler had made a mistake in his estimate of the ecliptic plane which resulted in the shift of the catalogue ecliptic pole into point P_A which differs from $P(t_A)$. Should this result in the perpendicular between P_A and the ecliptic pole motion trajectory cross it in point $t^* > t_A$ as is the case in fig. 5.5, this error of the compiler shall obviously add extra age to the catalogue ecliptic, namely, make it correspond best to the ecliptic of the year t^*. The opposite happens if this perpendicular crosses the trajectory in point $t^* < t_A$ – the author's mistake would thus make the catalogue "more recent". In order to give the reader an impression of the real value correlations, we shall indicate that for the Almagest the distance between ecliptic poles $P(0)$ for 1900 A.D. and $P(19)$ for the early A.D. period roughly equals 20' – the value approximates that of the error $\gamma_{stat}(t_A)$.

Earlier we mentioned that the value $\gamma_{stat}(t_A)$ equalled the length of segment $P(t_A)P_A$, whereas $\varphi_{stat}(t_A)$ was equal to angle $P_AP(t_A)D'(t_A)$. In a similar manner, $\gamma_{stat}(t) = \overline{P(t)P_A}$. Here the horizontal line on top refers to the length of the segment. However, angle $P_AP(t)D'(t)$ does not equal $\varphi_{stat}(t)$, since by moment t the vernal equinox axis would shift by the value $\omega(t_A - t)$.

Here ω stands for the annual angle velocity of precession that roughly equals 50", qv in Chapter 1. This shift corresponds to the value of angle $D'(t)P(t)S(t)$ in fig. 5.5. Thus, $\varphi_{stat}(t)$ is equal to the angle $P_AP(t)S(t)$, where angle $D'(t)P(t)S(t) = \omega(t_A - t)$.

In order to evade such bulky indications, let us assume that

$$x(t) = \overline{P(t)P(t_A)}, \quad y = \overline{P(t_A)P(t^*)}, \quad z = \overline{P_A P(t^*)},$$
$$\psi(t) = \angle P_A P(t) D'(t), \quad \delta = \angle D(t_A) P(t_A) P(t).$$

The value of $\gamma_{stat}(t_A)$ can be referred to as the ecliptic estimation error; it has the order of 20' in the Almagest. Angle δ does not depend on t and equals the angle between the motion direction of the ecliptic pole and line $D(t_A)D'(t_A)$ as estimated above. It is obvious that

$$z = \gamma_{stat}(t_A) \sin(\delta - \varphi_{stat}(t_A)),$$
$$y = \gamma_{stat}(t_A) \cos(\delta - \varphi_{stat}(t_A)).$$

Since $x(t) = v(t_A - t)$, from fig. 5.5 we get

$$\gamma_{stat}(t) = \sqrt{(v(t_A - t) + y)^2 + z^2} =$$
$$= \sqrt{\gamma^2_{stat}(t_A) + 2yv(t_A - t) + v^2(t_A - t)^2}. \tag{5.4.1}$$

Quite obviously, the minimal value of this function is reached with $t = t^*$. If we are studying a case of $|t - t_A| << |t_A - t^*|$, the function of $\gamma_{stat}(t)$ behaves almost as if it were linear:

$$\gamma_{stat}(t) \approx \gamma_{stat}(t_A) + v\cos(\delta - \varphi_{stat}(t_A))(t_A - t).$$

The function of $\varphi_{stat}(t)$ is also easy enough to find:

$$\varphi_{stat}(t) = \delta + \omega(t_A - t) - \arctan\left(\frac{z}{y + v(t_A - t)}\right). \tag{5.4.2}$$

Once again, if $|t - t_A| << |t_A - t^*|$, one can use linear approximation:

$$\varphi_{stat}(t) = \varphi_{stat}(t_A) + \left[\omega + \frac{v\sin(\delta - \varphi_{stat}(t_A))}{\gamma_{stat}(t_A)}\right](t_A - t).$$

Naturally, the formulae that we end up with can only give us a general idea of the character of such functions as $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$. In fig. 5.6 we can see an approximated representation of these functions that we get from the formulae 5.4.1 and 5.4.2. It is obvious that their actual form depends on the error rate

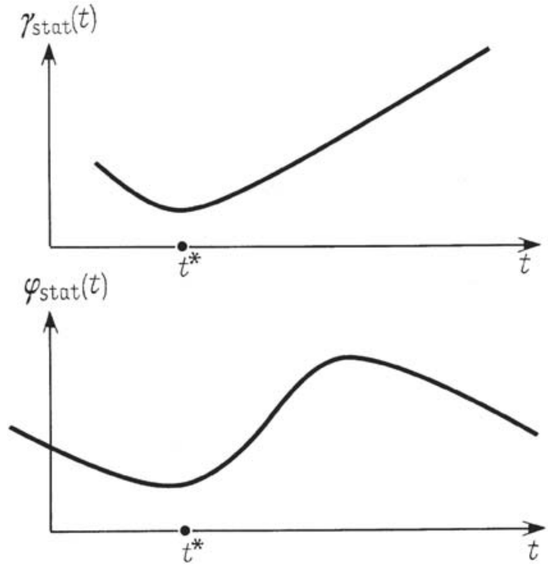


Fig. 5.6. Approximate view of the functions $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$.

for the catalogue compiler's accuracy, that is, the values of $\gamma_{stat}(t_A)$ and $\varphi_{stat}(t_A)$. Formulae 5.4.1 and 5.4.2 also define the nature of the dependency $\beta_{stat}(t)$, qv in formula 5.2.1.

Let us discuss the geometrical meaning of these calculations. We shall consider the Ptolemian coordinates of a certain star groups considering the observations to have been carried out in the time moment t. We must then compensate the systematic error $\gamma_{stat}(t)$, $\varphi_{stat}(t)$, or rotate the entire group by angle $\gamma_{stat}(t)$ around the axis which is on the distance $\varphi_{stat}(t)$ from the equinox axis. We shall assume that we have been perfectly precise in our estimation of the systematic error. Then the catalogue ecliptic pole P_A shall become superimposed over the real pole $P(t)$. Obviously, such a superimposition will not make the latitudinal discrepancies of the stars equal zero, since the catalogue also contains random errors. However, these errors do not affect the position of the ecliptic pole, having a null average value – or, rather, they affect it to a very small extent which is inversely proportional to the quantity of the star group under study.

From fig. 5.5 we see that the shift of pole P_A towards the point $P(t)$ can be decomposed into a composition of two shifts – P_A to $P(t_A)$ and $P(t_A)$ to $P(t)$

in a single possible manner. The parameters $\gamma_{stat}(t_A)$ and $\varphi_{stat}(t_A)$ which define the first shift refer to the observer's error, namely, the error made by the catalogue compiler in the estimation of the ecliptic plane. The second shift is defined by the centenarian fluctuation of the ecliptic plane which can be calculated by Newcombe's theory.

All of the above also implies the following corollary. Let us mark the latitudinal discrepancy of star i calculated for the presumed observation moment t as $\Delta B_i(t)$, and the same discrepancy for moment t after the compensation of the systematic error as $\Delta B_i^0(t) = \Delta B_i(t) - \gamma_{stat}(t)\sin(L_i(t) + \varphi_{stat}(t))$. Then the values of $\Delta B_i^0(t)$ shall be independent from t and equal the random errors made by Ptolemy in the estimation of the latitudes. The situation changes when mobile stars enter the stellar group under study. For them the value of $\Delta B_i^0(t)$ shall depend on the time t. The dependency character is defined by the values of individual random errors as well as the direction of proper motion velocities of all stars as viewed at once. In particular, for the unknown epoch t_A the value of $\Delta B_i^0(t_A)$ shall equal the random latitudinal error for star i. It would be natural to expect that if this star moves fast enough, and happens to be well-measured at the same time, the value of $\Delta B_i^0(t)$ should reach its minimum somewhere around the point t_A. The size of this minimum range depends on the value and the velocity of a given star's proper motion and equals hundreds of years even for the fastest of stars – Arcturus, for instance.

The above consideration and fig. 5.5 have a rather important implication that in order to determine the pole P_A of the catalogue ecliptic we only need to know the two values of γ_{stat} which will correspond to two respective time moment values – t_1 and t_2.

Indeed, Newcomb's theory makes it relatively easy to determine the ecliptic pole motion speed v, qv in Chapter 1. Let us fix two arbitrarily chosen time moments t_1 and t_2 (see fig. 5.7). We shall use the formula 5.3.7 to calculate the values of $\gamma_{stat}(t_1)$ and $\gamma_{stat}(t_2)$. Let us now draw the line of the ecliptic pole's motion through time, marking the points t_1 and t_2 thereupon. The scale we have to choose must make the distance between the two points equal $v|t_2 - t_1|$. The position of the ecliptic pole P_A is determined as the intersection point of the two circumferences whose centres

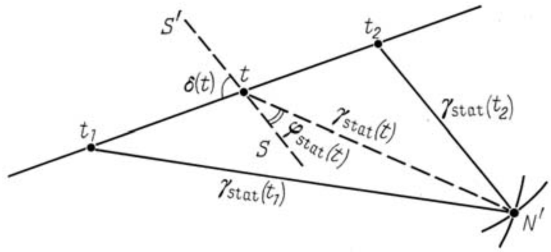


Fig. 5.7. Calculating the values of $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$.

are located in points t_i and whose radiuses equal $\gamma_{stat}(t_i)$, $i = 1.2$. Fig. 5.7 demonstrates how one calculates the values of $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ that correspond to arbitrary t values. It just has to be noted that the line $S'S$ that angle $\varphi_{stat}(t)$ is counted from crosses the trajectory of the ecliptic pole motion at angle $\delta(t)$. This angle can also be calculated with the aid of Newcomb's theory. The astronomical meaning of the straight line $S'S$ is obvious enough – it is a "straightened out" part of a large circumference pertaining to the celestial sphere that crosses the ecliptic pole $P(t)$ of epoch t and is perpendicular (at $P(t)$) to another large circumference which also crosses $P(t)$ and the equinox point of epoch t.

In a similar way, the calculation of the parameters $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ for all the values of t shall require the knowledge of two values only – $\varphi_{stat}(t_1)$ and $\varphi_{stat}(t_2)$.

We shall however work with angle γ. It is a pithy value, being the error in the estimation of the tilt angle between the equatorial and the ecliptic plane. Let us point out that this angle can be fixed with the use of the armillary sphere, for instance. Therefore, the error γ inherent in the value of this angle may be an instrumental error of the armillary sphere, qv in Chapter 1. Thus, error γ arises in the course of astronomical observation naturally. Apart from that, the choice of γ for the representation of a parameter shall further receive statistical validation.

5. THE STATISTICAL PROPERTIES OF THE ESTIMATES OF γ_{stat} AND φ_{stat}

We shall now consider the problem of calculating the parameters γ and φ which define the systematic error of the catalogue as a problem of statistics. Let us assume the following for this purpose: the catalogue compiler introduced the systematic error at time moment t_A; said error is defined by parameters γ_A and φ_A. Apart from that, let us assume the latitude of each measured star to have been affected by the random perturbation ξ_i with a zero average as a result of the observation error, or $E\xi_i = 0$. It is presumed that random errors ξ_i which correspond to different stars are independent and distributed uniformly. Let $\sigma^2 = E\xi_i^2$ stand for the dispersion of the random value ξ_i; this dispersion remains unknown to us in general.

The latitude of star i shall assume the following form in these presumptions:

$$b_i = B_i(t_A) - \gamma_A \sin(L_i(t_A) + \varphi_A) + \xi_i \qquad (5.5.1)$$

From the statistical point of view, what we have in front of us is a sample that consists of N realizations of random values $\{b_i\}_{i=1}^{N}$ of the 5.5.1 variety. This sample has to be used for the statistical calculation of $\hat{\gamma}$ and $\hat{\varphi}$ parameters of γ_A and φ_A, as well as the calculation of the σ value which is equal to the square average equation error. We shall localize the problem immediately and study the estimations of $\hat{\varphi} = \varphi_{stat}$ and $\hat{\gamma} = \gamma_{stat}$ calculated with the minimal square method. These estimations have the form of 5.3.6 and 5.3.7. Most of our attention shall be turned towards the estimation of the γ_A value for reasons explained at the end of Section 4.

Formula 5.5.1 looks traditional for regression analysis. Indeed, this equation claims observation error $\Delta b_i = B_i(t_A) - b_i$ to be a random value with the average $\gamma_A \sin(L_i(t_A) + \varphi_A)$ depending on unknown parameters γ_A and φ_A, and the dispersion σ^2. One has to estimate the values of unknown parameters using the minimal square method and determine the statistical qualities of the estimations received. Under such conditions, the curve $Y(x) = \gamma_A \sin(x + \varphi_A)$ is usually referred to as the line of regression.

Let us define the values of γ and φ using the relations expressed in 5.3.6 and 5.3.7. Discrepancies Δb_i are random by presumption. Therefore, the estimates of φ_{stat} and γ_{stat} that we get from these formulae are random values as well. Let us study their statistical qualities and consider their relation to the unknown true values of φ_A and γ_A.

Let us perform a substitution for s_b and c_b in the formulae related above, using the difference $\gamma_A \sin(L_i(t_A) + \varphi_A) - \xi_i$ instead of Δb_i and apply said substitution to formulae 5.3.6 and 5.3.7. We shall come up with the following expressions for the values φ_{stat} and γ_{stat}.

$$\tan\varphi_{stat} = \frac{\tan\varphi_A + \dfrac{\frac{1}{N}\sum_{i=1}^{N}\xi_i(s_2\cos L_i(t_A) - d\sin L_i(t_A))}{\gamma_A(d^2 - s_2c_2)\cos\varphi_A}}{1 + \dfrac{\frac{1}{N}\sum_{i=1}^{N}\xi_i(c_2\sin L_i(t_A) - d\cos L_i(t_A))}{\gamma_A(d^2 - s_2c_2)\cos\varphi_A}} \qquad (5.5.2)$$

$$\gamma_{stat} = \gamma_A - \frac{\frac{1}{N}\sum_{i=1}^{N}\xi_i(\sin L_i(t_A) + \tan\varphi_A\cos L_i(t_A))}{(s_2 + 2d\tan\varphi_A + c_2\tan^2\varphi_A)\cos\varphi_A}. \qquad (5.5.3)$$

Let us introduce the value

$$R = (\gamma_A(d^2 - s_2c_2)\cos\varphi_A)^{-1}.$$

In this case 5.5.2 can be transcribed as

$$\tan\varphi_{stat} = \frac{\tan\varphi_A + \frac{R}{N}\sum_{i=1}^{N}\xi_i(s_2\cos L_i(t_A) - d\sin L_i(t_A))}{1 + \frac{R}{N}\sum_{i=1}^{N}\xi_i(c_2\sin L_i(t_A) - d\cos L_i(t_A))}. \qquad (5.5.4)$$

The condition $E\xi_i = 0$ tells us that the received estimation of parameter γ_{stat} is not shifted, that is:

$$E\gamma_{stat} = \gamma_A. \qquad (5.5.5)$$

The dispersion for the estimation of γ_{stat} expressed through D_γ looks like this:

$$D_\gamma = \frac{\sigma^2}{N(\cos^2\varphi_A s_2 + 2d\cos\varphi_A\sin\varphi_A + c_2\sin^2\varphi_A)}. \qquad (5.5.6)$$

If observation errors ξ_i are distributed normally, the same applies to the value γ_{stat}, and the first two moments (5.5.5 and 5.5.6) define its entire distribution. This fact shall give us an opportunity to build the trust interval for the value of γ_A.

The estimation analysis of φ_{stat} is a bit more complex. Let us used the equation rendered from formula 5.5.4:

$$\tan\varphi_{stat} - \tan\varphi_A =$$

$$= \frac{\frac{R}{N}\sum_{i=1}^{N}\xi_i((s_2 + d\tan\varphi_A)\cos L_i(t_A) - (d + c_2\tan\varphi_A)\sin L_i(t_A))}{1 + \frac{R}{N}\sum_{i=1}^{N}\xi_i(c_2\sin L_i(t_A) - d\cos L_i(t_A))} \tag{5.5.7}$$

as well as the fact that for large values of N the second item in the denominator of the right part of 5.5.7 is a small value. This value is indeed of a random nature, with a null average and the dispersion of

$$\frac{\sigma^2 c_2}{N\gamma_A^2(s_2c_2 - d^2)\cos^2\varphi_A}.$$

If ξ_i are distributed normally, the same applies to the value under study. It has the following implication for the Almagest: even for $N = 30$ the probability P_N that the denominator of the right part of 5.5.7 shall be negative does not exceed 5×10^{-3}. This probability diminishes drastically with the growth of N: $P_{50} \le 2.5 \times 10^{-4}$, $P_{80} \le 4 \times 10^{-6}$, $P_{100} \le 3 \times 10^{-7}$, $P_{200} \le 8 \times 10^{-13}$, $P_{300} \le 2.5 \times 10^{-8}$.

Formula 5.5.7 implies that, in general, $E\tan\varphi_{stat} \neq \tan\varphi_A$. However, we can easily obtain distribution function $F(x)$ of the random value $\tan\varphi_{stat} - \tan\varphi_A$ from this formula which we need for the estimation of the trust interval for φ_A. Indeed, if we are to disregard the rather improbable case of the denominator in 5.5.7 becoming negative, we can educe the expression for $F(x)$ from this formula:

$$F(x) = P(\tan\varphi_{stat} - \tan\varphi_A < x) = P(\eta_x - x),$$

where random value η_x has the form of

$$\eta_x = \frac{R}{N}\sum_i \xi_i((s_2 + d(\tan\varphi_A + x)\cos L_i(t_A)) - (d + c_2(\tan\varphi_A + x)\sin L_i(t_A))).$$

Therefore, if values ξ_i are distributed normally with the dispersion equalling σ^2, value η_x shall have Gaussian distribution with a null average and the dispersion of

$$D(\eta_x) = \frac{R^2\sigma^2}{N}(c_2s_2 - d^2)(s_2 + 2d(x + \tan\varphi_A) + c_2(x + \tan\varphi_A)^2). \tag{5.5.8}$$

Thus,

$$F(x) = \Phi(x/\sqrt{D(\eta_x)}), \tag{5.5.9}$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du.$

The values of γ_{stat} and φ_{stat} as calculated above are the so-called punctual estimations of the unknown parameters γ_A and φ_A. Since we have found the distribution functions for these estimations, one can study the issue of possible errors inherent therein. Let us answer this question in standard terms used for trust intervals based on formulae 5.5.5, 5.5.6, 5.5.8 and 5.5.9.

In mathematical statistics the problem of confidence interval calculation is dependent on the following situation that we shall illustrate with the example of estimating the value of γ_A. This value is a deterministic error of a very certain nature made by the compiler of the catalogue. As a result of the statistical estimation of γ_A – with the aid of the minimal square method in our case – we end up with the random value γ_{stat}. One wonders about the boundaries of the unknown value γ_A if we already managed to determine γ_{stat}.

In order to keep these boundaries from becoming trivial, we have to define the acceptable error rate probability – that is, the probability of specifying such boundaries that shall not contain the true value of γ_A. Let us use ε for referring to the acceptable error rate probability. Confidence level shall equal $1 - \varepsilon$ in such a case. The random value of γ_{stat} is distributed normally, with parameters defined by formulae 5.5.5 and 5.5.6. Therefore, for $x > 0$ we shall have

$$P(|\gamma_{stat} - \gamma_A| < x) = \Phi(\sqrt{D_\gamma}x) - \Phi(-\sqrt{D_\gamma}x).$$

Let us define the value of $(\varepsilon/2)$ – the fractiles of normal x_ε distribution from the equation:

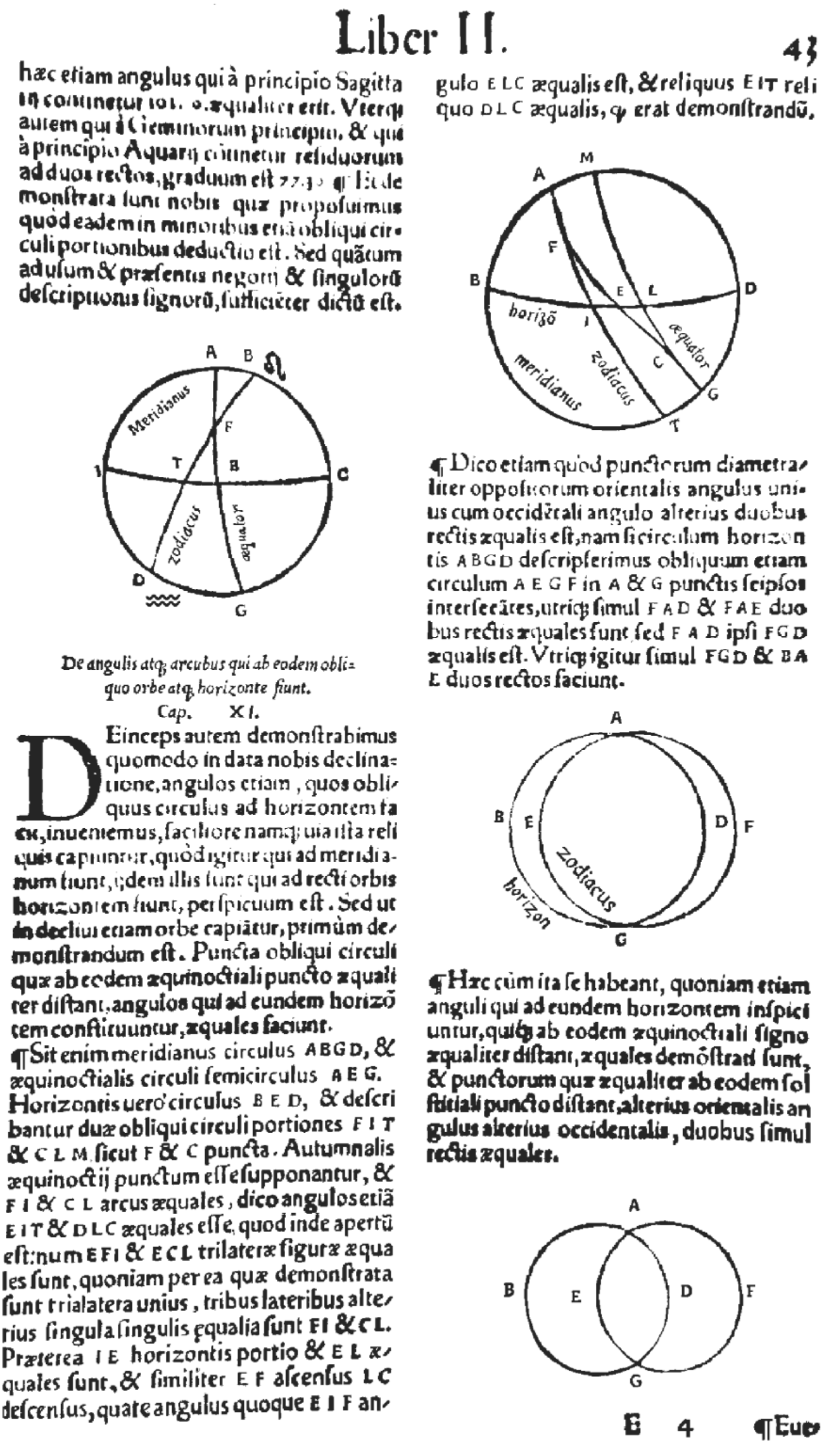
$$\Phi(\sqrt{D_\gamma}x_\varepsilon) - \Phi(-\sqrt{D_\gamma}x_\varepsilon) = 1 - \varepsilon,$$

or, alternatively, another equation that gives the same result $\Phi(-\sqrt{D_\gamma}x_\varepsilon) = \varepsilon/2$.

Then the interval

$$I_\gamma(\varepsilon) = (\gamma_{stat} - x_\varepsilon, \gamma_{stat} + x_\varepsilon) \tag{5.5.10}$$

shall represent the confidence interval for γ_A with confidence level of $1 - \varepsilon$. This follows from $P(|\gamma_{stat} - \gamma_A| \ge x_\varepsilon) = \varepsilon$.



Liber II. 43

hæc etiam angulus qui à principio Sagitta rij continetur 101. 0. æqualiter erit. Vterqȝ autem qui à Geminorum principio, & qui à principio Aquarij continetur residuorum ad duos rectos, graduum est 77.30 ¶ Et de monstrata sunt nobis quæ proposuimus quòd eadem in minoribus etiā obliqui circuli portionibus deductio est. Sed quātum ad usum & præsentis negotij & singulorū descriptionis signorū, sufficiēter dictū est.

De angulis atqȝ arcubus qui ab eodem obliquo orbe atqȝ horizonte fiunt.
Cap. XI.

Deinceps autem demonstrabimus quomodo in data nobis declinatione, angulos etiam, quos obliquus circulus ad horizontem facit, inueniemus, facilìore namqȝ uia ita reliquis capiuntur, quòd igitur qui ad meridianum fiunt, ijdem illis sunt qui ad recti orbis horizontem fiunt, perspicuum est. Sed ut in decliui etiam orbe capiātur, primùm demonstrandum est. Puncta obliqui circuli quæ ab eodem æquinoctiali puncto æqualiter distant, angulos qui ad eundem horizōtem constituuntur, æquales faciunt.

¶ Sit enim meridianus circulus A B G D, & æquinoctialis circuli semicirculus A E G. Horizontis uerò circulus B E D, & describantur duæ obliqui circuli portiones F I T & C L M, sicut F & C puncta. Autumnalis æquinoctij punctum esse supponantur, & F I & C L arcus æquales, dico angulos etiā E I T & D L C æquales esse, quod inde apertū est: num E F I & E C L trilateræ figuræ æquales sunt, quoniam per ea quæ demonstrata sunt trialatera unius, tribus lateribus alterius singula singulis equalia sunt E I & C L. Præterea I E horizontis portio & E L æquales sunt, & similiter E F ascensus L C descensus, quate angulus quoque E I F angulo E L C æqualis est, & reliquus E I T reliquo D L C æqualis, quod erat demonstrandū.

¶ Dico etiam quòd punctorum diametraliter oppositorum orientalis angulus unius cum occidētali angulo alterius duobus rectis æqualis est, nam si circulum horizontis A B G D descripserimus obliquum etiam circulum A E G F in A & G punctis seipsos intersecātes, utriqȝ simul F A D & F A E duobus rectis æquales sunt sed F A D ipsi F G D æqualis est. Vtriqȝ igitur simul F G D & B A E duos rectos faciunt.

¶ Hæc cùm ita se habeant, quoniam etiam anguli qui ad eundem horizontem inspiciuntur, quiqȝ ab eodem æquinoctiali signo æqualiter distant, æquales demōstrati sunt, & punctorum quæ æqualiter ab eodem solstitiali puncto distant, alterius orientalis angulus alterius occidentalis, duobus simul rectis æquales.

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Fig. 5.8. A page from a 1551 edition of the Almagest.

When we try to calculate the value of x_ε, we must particularly lean upon the value of D_γ, which depends on the unknown parameters σ^2 and φ_A. As it is usually done in mathematical statistics, we shall replace σ^2 in the formula for D_γ by the residual dispersion

$$\sigma_0^2(\gamma_{stat}, \varphi_{stat}, t_A) = \frac{1}{N}\sum_{i=1}^{N}(\Delta B_i(t_A) - \gamma_{stat}\sin(L_i(t_A) + \varphi_{stat}))^2,$$

defined by formula 5.5.3, and φ_A by φ_{stat}. Catalogue compilation moment t_A also remains unknown to us; thus, all the calculations as listed above have to be carried out for all the time moments t in order to estimate the systematic error $\gamma_{stat}(t)$, $\varphi_{stat}(t)$, assuming the catalogue to have been compiled in the random fixed epoch t.

In a similar way we can educe the confidence interval for φ_A with the confidence level of $1 - \varepsilon$. This interval $I_\varphi(\varepsilon)$ shall look like this:

$$\left(\varphi_{stat} - \frac{y_\varepsilon}{1+\tan^2\varphi_{stat} - y_\varepsilon\tan\varphi_{stat}},\quad \varphi_{stat} + \frac{y_\varepsilon}{1+\tan^2\varphi_{stat} + y_\varepsilon\tan\varphi_{stat}}\right), \tag{5.5.11}$$

y_ε being the solution of the equation $F(y) - F(-y_\varepsilon) = 1 - \varepsilon$, where distribution function F is defined by the equality 5.5.9, that is, $\varepsilon/2$ – fractile of the corresponding normal distribution.

Note: the above estimations of the true error rates for γ and φ in the catalogue as the presumed dating functions are not only important for our being able to compensate them, but also for the indirect verification of just how correct the suggested approach happens to be. For instance, if we came up with such a value of γ_{stat} that would be several times greater than the catalogue precision rate, it would indicate at the existence of substantial effects that we did not take into account.

However, inasmuch as the dating itself is concerned, the actual value of γ_{stat} takes no part in the corresponding procedure. All we need to know is the length of the respective trust interval. Therefore, one could simplify the calculations to a great extent in the following manner. One would have to calculate γ_{stat} and φ_{stat} for any fixed moment in time t_0: 1900 A.D., for instance, which would render Newcomb's calculations unnecessary. Then instead of the curves $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ we shall have constant values corresponding to observation errors – however, the coordinate system shall pertain to the epoch of 1900 A.D. Then we would draw confidence intervals around these constant values whose length will not depend on t. We shall end up with the same interval of possible catalogue datings as we did in our estimation of errors γ and φ for the presumed dating epoch t if we carry out the statistical dating procedure described below. The only information we shall lose after this shall be the estimated real values of γ_{stat} and φ_{stat}.

6. COROLLARIES

Corollary 1. The group error of a stellar configuration results in said configuration shifting across the celestial sphere as a whole. This shift can be parameterized by two parameters, namely, γ and φ (or γ and β), if we are to consider latitudinal discrepancies exclusively.

Corollary 2. The latitudinal discrepancies inherent in the catalogue can be reduced as a result of compensating the group errors.

Corollary 3. If group errors coincide for a large part of the catalogue, this common error is called systematic and can be discovered by statistical methods.

Under the condition that the catalogue compilation epoch equals t, the values of parameters $\varphi(t)$ and $\gamma(t)$ can easily be assessed with the minimal square method. The corresponding estimations of $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$ have the respective forms of 5.3.6 and 5.3.7.

Corollary 4. It suffices to know the values of $\gamma_{stat}(t_1)$ and $\gamma_{stat}(t_2)$ for two different moments in time for the reconstruction of functions $\gamma_{stat}(t)$ and $\varphi_{stat}(t)$.

Corollary 5. Confidence intervals $I_\varphi(\varepsilon)$ and $I_\gamma(\varepsilon)$ for the real values of parameters $\varphi(t)$ and $\gamma(t)$ were calculated under the assumption of random errors being distributed normally. See the respective formulae 5.5.11 and 5.5.10.

Let us conclude by reproducing a page from a 1551 edition of the Almagest in fig. 5.8.