$a_{rs} = L(\Delta_r, \Delta_s)$ with the number of common names $0(\Delta_r, \Delta_s) = \text{const}$, being fixed. For the lists P and N, all these histograms were of the form shown in Fig. 9. The list value was taken as the threshold to the right of which the histogram was vanishing. The relations exceeding the threshold value are termed below as essential. Note that all the disjoint neighbourhoods for the P and N as expected, turned out to be dependent according to the constructed test (i.e. their relations were essential).

15 The results of numerical analysis

Here and in the next items, we consider the consequences of the study of the relation matrix for the popes' lists, from Peter until 1950, and Roman emperors from the 4th century A.D., Western Roman Empire until the Holy Roman Empire in 962–1254 and the Habsburgh Empire in 1273–1619 A.D., the emperors' list extended up to 1700 A.D. (according to Leopold). To make the discussion of the results independent of the argument of §§ 13 and 14 above, we recall the basic idea of the method.

The so-called *relation matrix* is constructed from a large chronological list of rulers' names, for which each pair of connected fixed-length pieces (neighbourhoods) is associated with a number (relation), so that the following conditions are fulfilled, namely, in the case where the given list contains no duplicates, and consists of a random (in a sense) name sequence, the mean value of the relation does not depend on the choice of the numbers of the neighbourhood pairs, and, in the case where the list does contain duplicates, the relation of the pairs possessing duplicating fibers is, generally speaking greater than for those without such fibers.

By examination of the matrix of the distribution frequency for a fixed number of common names in the neighbourhoods, thresholds were defined, separating the essential relations (the conclusion being that the neighbourhoods do contain duplicating fibers) and inessential relations (with the conclusion that the neighbourhoods are independent); see Fig. 9.

The *essential relation matrix* (or simply *relation matrix*) thus obtained provides for a decomposition of the list into duplicate systems, (meanwhile, different systems can intersect, i.e. certain parts of the list possibly 'fiber'). We note briefly certain overlappings determined by the constructed matrices:

(i) three sharp peaks of heresies in church history: the 4th century, 'heresy age'; 1st to 12th centuries A.D., the Albigenses war with the heretics, establishment of the Inquisition; and heresies of the 13th to 16th centuries, strengthening of the Inquisition, putting witches to death, religious wars (Fig. 10(a));
(ii) church schism in 1054 A.D.; irrevocable separation of the church in 1204 A.D. (Fig. 10(a));
(iii) three papal elections under Henry III around 1050 A.D.; the three popes in 1378–1417 A.D. during the Great Schism (Fig. 10(a));
(iv) Roman Empire, 753–523 B.C.; Roman Empire, 82 B.C.–217 A.D. (Fig. 10(b));
(v) Roman Empire, 82 B.C.–217 A.D.; Roman Empire, 270–526 A.D.; Holy Roman Empire, 962–1254 A.D.; Roman Empire, 270–526 A.D.; Holy Roman Empire, 962–1254 A.D. (Fig. 10(b));
(vi) Carolingians, 681–887 A.D.; the Holy Roman Empire, 962–1254 A.D.; and the Habsburg Empire, 1273–1619 A.D.;
(vii) Overlapping of the limits of the two Roman republics (Fig. 10(b)).

The popes' list for 50–1950 A.D. contains 87 different names in a total of 293, their maximum multiplicity being 21. The essential relation matrix is shown in Fig. 10(a). Note